

SHORT ROPES AND LONG KNOTS

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We study spaces of non-singular smooth embeddings of a closed interval into \mathbf{R}^3 . Our main result is a geometric interpretation of the Grothendieck group of the monoid of knots.

1. ROPES

Let us begin with definitions. Fix two points A and B in \mathbf{R}^3 . We shall always assume that $A = (0, 0, 0)$ and $B = (1, 0, 0)$ so, in particular, the line AB is the x -axis and the length of the interval $[AB]$ is equal to 1. (We will write $[AB]$ for the closed interval between A and B , (AB) for the corresponding open interval, and AB for the line passing through A and B .)

A *rope* is a non-singular C^1 -smooth embedding $r : [0, 1] \rightarrow \mathbf{R}^3$ such that $r(0) = A$ and $r(1) = B$. Non-singularity here includes the condition that the tangent vectors $\frac{dr}{dt}(0)$ and $\frac{dr}{dt}(1)$ are non-zero.

The space of all ropes endowed with C^1 -topology is denoted by B_∞ . We will also consider, for any $\epsilon > 0$, its subspaces B_ϵ that are formed by ropes whose length is strictly less than $1 + \epsilon$. Each of these subspaces comes with a natural basepoint — the tight rope which is the embedding $t \rightarrow (t, 0, 0)$. Finally, we say that a rope is *short*, if its length is less than 3.

Even though ropes are knotty objects, spaces of ropes B_ϵ have only one connected component. Indeed, any knotted rope can be “undone”, i.e. deformed to the tight rope without increasing its length. (Figure 1 (f)-(i) illustrates how a knotted rope can be undone.) However, the fundamental group of B_ϵ can be non-trivial and we shall see that, for spaces of short ropes, it is closely related to the semigroup of knots.

Unless stated otherwise, by “knots” we will mean “long” or “non-compact” knots, i.e. nonsingular smooth embeddings $\mathbf{R} \rightarrow \mathbf{R}^3$ whose tangent vectors tend to $(1, 0, 0)$ at $\pm\infty$. These are essentially knots in S^3 and their isotopy classes are in one-to-one correspondence with the isotopy classes of “round” knots. Recall that the isotopy classes of knots form a commutative monoid (i.e. a semigroup with a unit) K under the connected sum. The monoid K is freely generated by classes of prime knots, of which there are countably many; the unknot, that is, the inclusion map of the x -axis into \mathbf{R}^3 , being the unit. We will often say “knots” for both geometric objects and their isotopy classes, it will always be clear from the context which we are talking about.

To each $k \in K$ we can associate an element $b_\epsilon(k) \in \pi_1(B_\epsilon)$ for any ϵ as follows. First by carrying the rope around the point A we tie the knot k on the rope near A . Then we push the knot along the rope all the way to the point B and throw it off the rope there. (See Figure 1.) This process defines a closed path in B_ϵ and as we will see later the homotopy class of this path corresponds to a well-defined element of $\pi_1(B_\epsilon)$. The precise definition of the map b_ϵ will be given in Section 3.

It is easy to see that the map b_ϵ respects the connected sum of knots; that is to say that it is a homomorphism of the monoid K to the group $\pi_1(B_\epsilon)$.

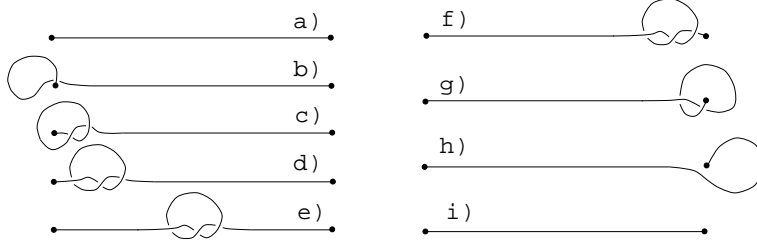


FIGURE 1. The image of a trefoil under b_ϵ .

Our main result is the following

Theorem 1.1. *For any $0 < \epsilon \leq 2$ the homomorphism $b_\epsilon : K \rightarrow \pi_1(B_\epsilon)$ is a group completion.*

In other words, for any space of short ropes B_ϵ the group $\pi_1(B_\epsilon)$ is the Grothendieck group \widehat{K} of the monoid K . It is easy to understand which loop corresponds to $-k \in \widehat{K}$: it is defined in the same way as $b_\epsilon(k)$, but we tie the knot at B and push it to the left towards A .

Interestingly, the situation changes dramatically as soon as ϵ becomes bigger than 2.

Theorem 1.2. *For any $\epsilon > 2$ the space B_ϵ is simply-connected.*

The reason for such behaviour can be roughly explained as follows. To undo a knot which is tied on a short rope one needs to carry the rope either around A or B . However, if the rope is longer than 3, we can take the knot off the whole interval $[AB]$, see Figure 2.

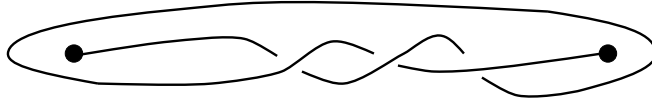


FIGURE 2. A rope which is not short.

At the moment we cannot say anything about the higher homotopy or homology groups of B_ϵ for any finite ϵ . However, in the limit case $\epsilon = \infty$ an explicit deformation retraction of B_∞ to the tight rope can be constructed. Thus we have the following:

Theorem 1.3. *The space of all ropes B_∞ is contractible.*

Our last result says that all spaces of short ropes have the same homotopy type:

Theorem 1.4. *The natural inclusion $B_\epsilon \hookrightarrow B_{\epsilon'}$ is a homotopy equivalence for all $0 < \epsilon \leq \epsilon' \leq 2$.*

This is all we know about ropes so far. This work is far from being a conclusive study and some questions are listed in the last section. The next section briefly explains where the idea of studying ropes comes from and in sections 3, 4, 5 and 6 we prove Theorems 1.1, 1.2, 1.3 and 1.4 respectively.

Much of the inspiration for this work came from discussions with Sergei Chmutov, Sergei Duzhin, Elmer Rees and Simon Willerton who has also given some valuable comments on the preliminary version of this paper.

2. MOTIVATION: CLASSIFYING SPACES

The motivation for studying ropes comes from the construction of a classifying space for a topological monoid, described by M.C.McCord in [6]. Consider a space BM whose points are configurations of particles on the interval $[0, 1]$ and all particles are labelled by non-zero elements of a monoid M . The topology is introduced in such a way that it agrees with the topologies on $[0, 1]$ and the label space M ; if two particles with coordinates x_1 and x_2 (where $x_1 < x_2$) and labels h_1 and h_2 respectively move towards each other and eventually collide, they form a particle with the label $h_1 h_2$. The endpoints of the interval are “sources” of particles, i.e. particles with arbitrary labels can appear from 0 and 1 and, conversely, when a particle collides with 0 or 1 it disappears. The basepoint in BM is chosen to be the empty configuration. (See [6] for the precise definition.)

Theorem 2.1. [6] *For M a simplicial monoid, BM is a classifying space of M .*

Remark. McCord’s theorem as stated in [6] deals with classifying spaces of groups rather than monoids. McCord’s construction, however, is exactly the same thing as Segal’s construction of a classifying space for a category [9] in the case when the category has only one object.

An example of this construction is the infinite symmetric product of a circle $SP^\infty(S^1)$. Here particles are labelled by positive integers and labels are just the multiplicities of points. The particles of $SP^\infty(S^1)$ live on a circle rather than on an interval; notice, however, that in McCord’s construction points 0 and 1 can be identified. The infinite symmetric product of S^1 is well-known to be topologically a circle, and this is the classifying space for the monoid of non-negative integers.

If G is a discrete group there is a map $\omega : G \rightarrow \Omega BG$ which induces an isomorphism $G \rightarrow \pi_1(BG)$. The image of $g \in G$ under ω is the loop which at time t is a configuration with a single particle which has coordinate t and label g . If M is a discrete abelian monoid the map ω is also defined in the same way and it induces the standard homomorphism of M to its Grothendieck group \widehat{M} . The latter statement is a particular case of the Homology Group Completion Theorem (see [1] or [7]). Clearly, if $m_1, m_2 \in M$, the element of $\pi_1(BM)$ which corresponds to the image of $m_1 - m_2$ in \widehat{M} can be represented by the loop $\omega(m_1)\overline{\omega}(m_2)$; here by $\overline{\omega}(m_2)$ we mean the loop $\omega(m_2)$ taken with the opposite parametrisation.

McCord’s construction can, of course, be applied to the monoid K of the isotopy classes of knots. The classifying space BK can be then thought of as an interval on which infinitesimally small knots are tied. Placing the interval into \mathbf{R}^3 and replacing infinitesimally small knots by knots of finite size we come to the notion of a rope; the map b_ϵ described in Section 1 is an analogue of the map ω . The condition $\epsilon \leq 2$ in this context means that knots that are tied on a rope can be “localised” as particles. This is the main idea of the proof of Theorem 1.1.

3. SPACES OF SHORT ROPES

We define an *extension* of a rope r as the map $\tilde{r} : \mathbf{R} \rightarrow \mathbf{R}^3$ which coincides with r on the interval $[0, 1]$ and such that $\tilde{r}(t) = (t, 0, 0)$ for all $t \in (-\infty, 0] \cup [1, \infty)$. An extension of a rope is a piecewise-smooth long knot which, however, can have points of self-intersection, see Figure 3. We say that an extension of a rope r has no singularities to the left of A if for all $t \in (-\infty, 0]$ the point $\tilde{r}(t)$ is not a point of self-intersection. Similarly one defines what it means for an extension of a rope to have no singularities to the right of B .

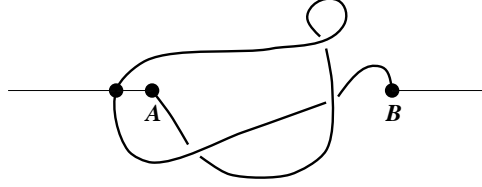


FIGURE 3. A knot extension of a rope.

Now we can give the precise definition of the map b_ϵ . An element of $\pi_1(B_\epsilon)$ can be represented by a loop on B_ϵ , that is, by a one-parameter family of ropes $L_T : [0, 1] \rightarrow B_\epsilon$ such that $L_0 = L_1 = \text{tight rope}$. We say that a family $L_T : [0, 1] \rightarrow B_\epsilon$ is *generic* if there is a finite number of values of the parameter T_i and T'_j such that the ropes L_{T_i} and $L_{T'_j}$ extend to knots with only one transversal double point to the left of A or to the right of B respectively, and for all other values of T the rope L_T extends to a genuine knot.

The map b_ϵ assigns to a knot k a one-parameter family of ropes $L_T : [0, 1] \rightarrow B_\epsilon$ which is generic in the above sense and such that:

- there exists $X \in [0, 1]$ for which $T_i < X < T'_j$ for all i, j ;
- the knot extension of L_X has isotopy class k ;
- $L_0 = L_1 = \text{tight rope}$.

Lemma 3.1. *The map $b_\epsilon : K \rightarrow \pi_1(B_\epsilon)$ is well-defined.*

To prove this it is enough to show that if a loop L_T on B_ϵ can be extended to the right of B without singularities for any $T \in [0, 1]$, then it is contractible in B_ϵ . This is an immediate corollary of Lemma 3.5 below.

Assuming the truth of Lemma 3.1, we shall now prove that the map \hat{b}_ϵ of the Grothendieck group \hat{K} into $\pi_1(B_\epsilon)$, which is induced by the map b_ϵ is an isomorphism for any $0 < \epsilon \leq 2$.

Proposition 3.2. *For $0 < \epsilon \leq 2$ the map $\hat{b}_\epsilon : \hat{K} \rightarrow \pi_1(B_\epsilon)$ is a monomorphism.*

Proof. Let p be the projection onto the line AB and for a rope r let $A(r) \subset AB$ be the subset of such points t that the inverse image of $p \circ r(t)$ in $[0, 1]$ consists of more than one point or that the tangent line to r at t is orthogonal to AB , see Figure 4 (a). We say that a rope r is *nice* if $A(r)$ is a union of finitely many closed intervals and points. For example, analytic ropes are nice.

Suppose r is a nice rope and $A(r) = \bigcup A_i(r)$, where $A_i(r)$ are disjoint closed intervals or points (here i belongs to some finite index set). To each $A_i(r)$ which

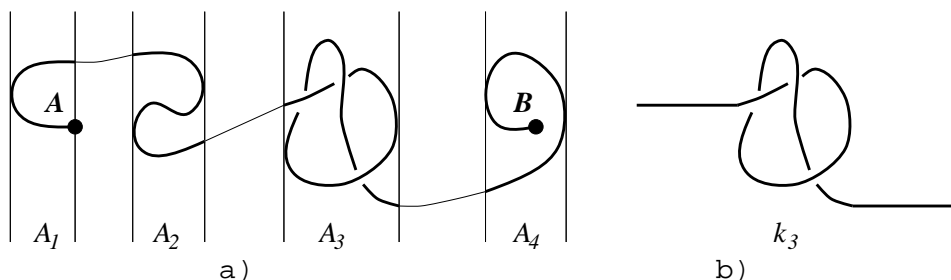


FIGURE 4. A nice rope and one of the corresponding knots.

does not contain A or B corresponds a knot $k_i(r)$ which is obtained by extending the segment of the rope r which projects onto $A_i(r)$ to the left and to the right by rays parallel to the line AB (see Figure 4 (b)). Clearly, if $A_i(r)$ is a point, the corresponding $k_i(r)$ is a trivial knot. We say that nice ropes r_1 and r_2 are of the same type if $A(r_1)$ can be identified with $A(r_2)$ by some orientation-preserving self-homeomorphism of AB which fixes A and B , and if knots that correspond to $A_i(r_1)$ and $A_i(r_2)$ are the same.

Recall that a point in BK is a collection of particles labelled by knots on a closed interval. Let us identify this interval with the interval $[AB]$. We say that a point $y \in BK$ is *subordinate* to a nice rope r if all particles of y belong to $A(r)$ and if the sum of the coefficients of particles that are contained in $A_i(r)$ is exactly $k_i(r)$. (If $A_i(r)$ contains A or B we allow the coefficients of particles within $A_i(r)$ to add up to any knot.)

Let $S(r) \subset BK$ be subspace of points that are subordinate to the rope r . Clearly, if r_1 and r_2 have the same type, $S(r_1)$ and $S(r_2)$ are homeomorphic. Notice that $\epsilon \leq 2$ implies that the intervals A_i do not cover the whole $[AB]$ and, hence, for any short rope r the space $S(r)$ is contractible.

Now suppose that a formal difference of knots $k_1 - k_2 \in \widehat{K}$ defines a contractible loop γ_T in B_ϵ . We can assume that γ_T goes through nice ropes and that the standard loop $\omega_T(k_1)\overline{\omega_T}(k_2)$ which represents $k_1 - k_2$ in BK (see Section 2) is subordinate to γ_T at any value of the parameter T . So there exists a map $F : [0, 1]^2 \rightarrow B_\epsilon$ which coincides with γ_T on one side of the square and sends the rest of its boundary to the tight rope.

Lemma 3.3. *Without loss of generality we can assume that:*

- the image of F consists of nice ropes only;
- the square $[0, 1]^2$ can be triangulated in such a way that the interiors of all simplices of triangulation are mapped to ropes of the same type.

The proof is purely technical. One can use Fourier expansions to replace the 2-dimensional family of ropes F by an analytic family of analytic ropes; in this context the lemma is easily verified. The details, in which neither God nor devil are to be found, are left to the reader.

Consider the subspace $S \subset [0, 1]^2 \times BK$ of all pairs $\{x, y\}$ where $x \in [0, 1]^2$ and $y \in S(F(x))$. Under the assumptions of Lemma 3.3 one can check that the projection $S \rightarrow [0, 1]^2$ is a quasifibration with contractible fibres. (For the definition and properties of quasifibrations see [3].) By the weak homotopy lifting property

of quasifibrations there is a map $F' : [0, 1]^2 \rightarrow S$ which sends $(T, 0) \in [0, 1]^2$ to $\{(T, 0), \omega_T(k_1)\overline{\omega_T}(k_2)\}$ and any other point x of the boundary to $\{x, *\}$; here $*$ $\in BK$ is the basepoint. Combining F' with the projection $S \rightarrow BK$ we see that $\omega_T(k_1)\overline{\omega_T}(k_2)$ is contractible in BK and this is possible only if $k_1 = k_2$. \square

Proposition 3.4. *For any $\epsilon > 0$ the map $\hat{b}_\epsilon : \hat{K} \rightarrow \pi_1(B_\epsilon)$ is an epimorphism.*

Proof. Let $W_L \subset B_\epsilon$ be the subspace of ropes which extend to knots without singularities to the left of the point A and let W_R denote the subspace of ropes extending without singularities to the right of B .

Lemma 3.5. *For any $\epsilon > 0$ the subspaces W_L and W_R are contractible.*

The proof of this lemma is rather technical and is better visualised than verbalised. We postpone it till the end of the section.

Ropes which lie in the intersection $W_L \cap W_R$ extend to non-singular (apart from a possible discontinuity of the tangent vectors at A and B) knots. So $\pi_0(W_L \cap W_R) = K$, as it is easy to check that the restriction on the length of the ropes does not influence the picture. As W_L and W_R are contractible, the union $W_L \cup W_R$ has the homotopy type of the suspension on $W_L \cap W_R$. Consequently, $\pi_1(W_L \cup W_R)$ is the free group, generated by the non-zero elements of K .

The complement of $W_L \cup W_R$ in B_ϵ has codimension 2, so the map

$$\pi_1(W_L \cup W_R) \rightarrow \pi_1(B_\epsilon)$$

induced by the inclusion $W_L \cup W_R \rightarrow B_\epsilon$ is onto. The problem now is to find the relations in $\pi_1(W_L \cup W_R)$.

Recall our definition of the map $b_\epsilon : K \rightarrow \pi_1(B_\epsilon)$. It is clear from the definition that each knot defines, in fact, a loop in $W_L \cup W_R$, as a rope in the process of deformation intersects the line AB first on the left of A and then on the right of B . So there is a well-defined map $K \rightarrow \pi_1(W_L \cup W_R)$ which, however, is not a homomorphism. One can easily identify the image of this map: a knot is mapped to the corresponding generator of the group $\pi_1(W_L \cup W_R)$.

We know that the composite map

$$K \rightarrow \pi_1(W_L \cup W_R) \rightarrow \pi_1(B_\epsilon)$$

is a homomorphism. In case $\epsilon \leq 2$ this immediately determines the set of relations we are looking for: the generators must commute as K is abelian, and $k_1 k_2$ (the product in $\pi_1(W_L \cup W_R)$) must be equal to the connected sum $k_1 \# k_2$. These relations define the Grothendieck group \hat{K} and, as we have seen above, for $\epsilon \leq 2$ the group $\pi_1(B_\epsilon)$ cannot be smaller. \square

In case $\epsilon > 2$ there may be some additional relations. And indeed, in the next section we shall see that in this case $\pi_1(B_\epsilon) = 0$.

Proof of Lemma 3.5. The spaces W_L and W_R are homeomorphic so it suffices to verify the statement of the lemma for W_L .

The deformation retraction of W_L to the tight rope is done in two stages. Let $W_L^0 \subset W_L$ be the subset of ropes whose tangent vector at A is $(a, 0, 0)$ for some $a > 0$; the extensions of all ropes in W_L^0 to the left can be parametrised so as to be smooth. First we will show how to deform W_L into W_L^0 and then prove that W_L^0 can be deformed to the tight rope.

Choose the spherical coordinates (r, ϕ, θ) in \mathbf{R}^3 with the centre at A and $\theta = 0$ being the positive half of the x -axis in \mathbf{R}^3 . Consider the family of maps

$$d_T : (r, \phi, \theta) \rightarrow (r, \phi, \theta \cdot (1 - \frac{5T}{6}))$$

with $T \in [0, 1]$. Away from the non-positive part of the x -axis and for each $T \in [0, 1]$ the map d_T is a diffeomorphism onto its image. Moreover, d_T does not increase lengths and thus we have the corresponding deformation of the space of ropes

$$D_T : W_L \times [0, 1] \rightarrow W_L.$$

The effect of the map D_1 is that all ropes in W_L are pushed into the cone $\theta < \pi/6$. In order to deform them further to W_L^0 we need to “squeeze” the tip of this cone, see Figure 5. The subtle point here is that it has to be done while keeping the lengths of ropes under control.

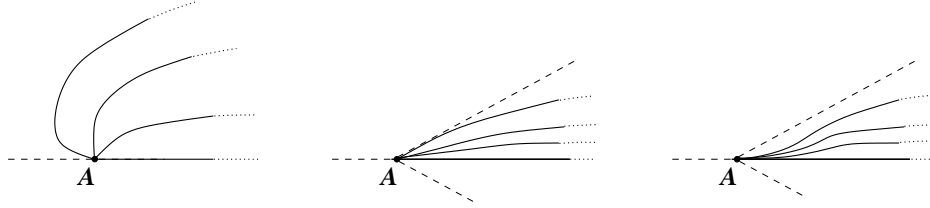


FIGURE 5. The deformation of W_L to W_L^0 near the point A .

Define the distance $\rho(r_1, r_2)$ between two ropes r_1 and r_2 by

$$\rho(r_1, r_2) = \max_{t \in [0, 1]} (|r_1(t) - r_2(t)| + |\frac{d}{dt}(r_1(t) - r_2(t))|).$$

Together with the Euclidean metric on \mathbf{R} the distance function ρ gives rise to a distance function on $B_\epsilon \times \mathbf{R}$. Identifying the line AB with \mathbf{R} in the obvious way, that is, $A = 0$ and $B = 1$, we obtain a distance function on $B_\epsilon \times AB$.

Let $E \subset B_\epsilon \times AB$ be the subspace of pairs (r, x) such that $(p \circ r)^{-1}(x)$ consists of only one point at which the angle between the tangent vector to r and the line AB is less than $\pi/4$. Notice that for any rope $r \in D_1(W_L)$ the pair $(r, 0)$ lies in E .

Define $\delta_1(r)$ to be the infimum of the distance between $(r, 0)$ and the complement of E in $B_\epsilon \times AB$, and let $\delta_2(r)$ be equal to $1 + \epsilon - l(r)$, where $l(r)$ is the length of the rope r . On the subspace $D_1(W_L)$ both $\delta_1(r)$ and $\delta_2(r)$ are continuous and strictly positive functions of r , so if we set

$$\delta(r) = \frac{1}{5} \min(\delta_1(r), \delta_2(r))$$

the function $\delta(r)$ is also positive and continuous on $D_1(W_L)$.

Let us fix a smooth function $f : [0, +\infty) \rightarrow \mathbf{R}$ such that $f(0) = 0$, $f(x) = 1$ for $x \geq 1$ and $0 < \frac{df}{dx} < 2$ for $x < 1$. For $r \in D_1(W_L)$ and $T \in [0, 1]$ consider the function $f_{r,T}(x) = Tf(\frac{x}{\delta(r)}) + (1 - T)$. Notice that $f_{r,T}(x)$ is continuous in r as $\delta(r)$ is positive and continuous on $D_1(W_L)$. Then the deformation retraction

$$D'_T : D_1(W_L) \times [0, 1] \rightarrow D_1(W_L)$$

can be defined on a rope $r = (r_x, r_y, r_z)$ as

$$D'_T(r) = (r_x, f_{r,T}(r_x)r_y, f_{r,T}(r_x)r_z).$$

The boundedness of $\frac{df}{dx}$ and the condition that $f(0) = 0$ imply that at $T = 1$ all ropes are carried into ropes whose tangent vectors point in the direction of the x -axis. It remains to check that $D'_T(r)$ does not increase the lengths of ropes too much.

Notice that under D'_T a rope r changes only near the point A , namely within the region $0 < x < \delta(r)$. Recall that by definition we have $\delta(r) < \delta_1(r)$ so for all $0 < x < \delta(r)$ the pair (r, x) belongs to E . This, in particular, means that for $0 < x < \delta(r)$ the rope r can be re-parametrised as $(x, r_y(x), r_z(x))$ with $(\frac{d}{dx}r_y(x))^2 + (\frac{d}{dx}r_z(x))^2 < 1$. It follows that the part of r within the region $0 < x < \delta(r)$ has length between $\delta(r)$ and $\delta(r)\sqrt{2}$.

Now,

$$\left| \frac{d}{dx}(f_{r,T}r_y) \right| \leq \left| \frac{d}{dx}(f_{r,T}) \right| |r_y| + |f_{r,T}| \left| \frac{d}{dx}r_y \right| \leq \left| \frac{2T}{\delta(r)} \right| |\delta(r)| + 1 \cdot 1 \leq 3$$

and the same inequality holds for $|\frac{d}{dx}(f_{r,T}r_z)|$. Hence, the length of the part of $D'_T(r)$ contained in the region $0 < x < \delta(r)$ is bounded by $\delta(r)\sqrt{1+3^2+3^2} < 5\delta(r)$. It follows that D'_T can only increase the length of a rope r by less than $5\delta(r) \leq \delta_2(r) = (1+\epsilon) - l(r)$ which means that the total length of $D'_T(r)$ is less than $1+\epsilon$ for all T and r .

Let us now describe the second stage, which is the deformation D''_T of W_L^0 to the tight rope. For each r we construct $D''_T(r)$ (with D''_1 being the identity map and D''_0 — the map to the tight rope) in several steps. Without loss of generality we will assume that all ropes are parametrised by length (up to a constant factor).

First we cut the rope r at the value of the parameter T , i.e. consider the embedding $r : [0, T] \rightarrow \mathbf{R}^3$. If the projection $p(r(T))$ of the point $r(T)$ onto AB lies to the right of the point B , we “squeeze the rope” linearly by a map $s : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ given by

$$(x, y, z) \rightarrow \left(\frac{x}{|p(r(T))|}, y, z \right)$$

so that $p(s(r(T))) = B$.

If $p(r(T))$ lies to the left of B we “move the rope to the right” by a translation s' , such that $p(s'(r(T))) = B$, see Figure 6. (The interval between A and $(1 - p(r(T)), 0, 0)$ is filled in with a segment of a straight line.) One can check that when $p(r(T)) \rightarrow B$ both s and s' tend to the identity map.

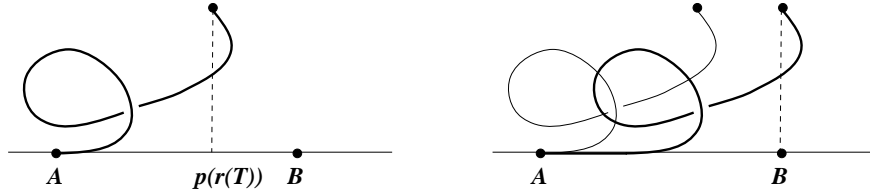


FIGURE 6. “Moving a rope to the right”.

One of the endpoints of the resulting curve is A ; let us denote the other endpoint by B' . Then the second step is a shift in the planes orthogonal to AB given by

$$(x, y, z) \rightarrow (x, y, z) - x^2 \cdot (B' - (1, 0, 0)).$$

After this shift the curve obtained is a rope r' (or, more precisely, an image of a rope) as the ends of it coincide with A and B respectively. The length of r' , however, may exceed $1 + \epsilon$ so we need the third step. It is a transformation induced by squeezing \mathbf{R}^3 to the line AB with the help of a continuous function $h_{r,T}$ such that $0 < h_{r,T} \leq 1$:

$$(x, y, z) \rightarrow (x, y h_{r,T}, z h_{r,T}).$$

The function $h_{r,T}$ is chosen to be equal to 1 if the length of r' is not greater than the length of r ; otherwise we define $h_{r,T}$ by the condition that the rope $(r'_x, r'_y h_{r,T}, r'_z h_{r,T})$ has the same length as r .

Finally, we parametrise the rope by length (up to a constant factor) and the desired deformation D_T'' is the composition of all the above transformations.

Now it can be checked directly that the map $D_T'' : W_L^0 \times [0, 1] \rightarrow W_L^0$ is continuous and that $D_0''(W_L^0)$ is the tight rope. \square

4. SPACES OF LONG ROPES

In the previous section we have seen that for any $\epsilon > 0$ the map $\hat{K} \rightarrow \pi_1(B_\epsilon)$ is onto. Here we will show that if $\epsilon > 2$ this map is a Vassiliev invariant of order 1. It is a well-known fact that all Vassiliev knot invariants of order 1 are constants; so this will imply that $\pi_1(B_\epsilon) = 0$ for $\epsilon > 2$.

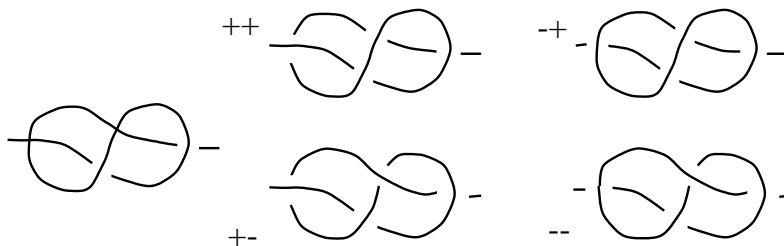


FIGURE 7. A singular knot k and knots k_{++}, k_{+-}, k_{-+} and k_{--} .

For basic facts about Vassiliev invariants we refer the reader to [2], [10] or [11]. Here we recall the definition of a Vassiliev invariant of order 1. In the space of all smooth maps $\mathbf{R} \rightarrow \mathbf{R}^3$ that coincide with a chosen line outside some finite interval there is a discriminant Δ which is formed by non-embeddings. The space of knots is then the complement of Δ in the space of all maps. The stratum Δ_1 of Δ which has codimension 1 is formed by knots with generic double points and the stratum Δ_2 of codimension 2 is formed by knots with 2 double points. A neighbourhood of a point on Δ_2 is pictured on Figure 8 (a). Here k_{++}, k_{+-}, k_{--} and k_{-+} are the knots obtained by resolving the singularities of a knot with two double points. An example of such resolution is given on Figure 7. A function

$$v_1 : K \rightarrow G$$

which takes values in some abelian group G is a Vassiliev invariant of order 1 if for all points of Δ_2

$$v_1(k_{++}) - v_1(k_{+-}) + v_1(k_{--}) - v_1(k_{-+}) = 0.$$

The same definition is valid for round knots, i.e. embeddings $S^1 \rightarrow \mathbf{R}^3$.

Any knot invariant can be extended from K to \widehat{K} by linearity. In particular, a Vassiliev invariant of order 1 on \widehat{K} is a linear extension of a Vassiliev knot invariant of the same order. It is clear that a knot invariant is identically zero if and only if its extension to \widehat{K} is.

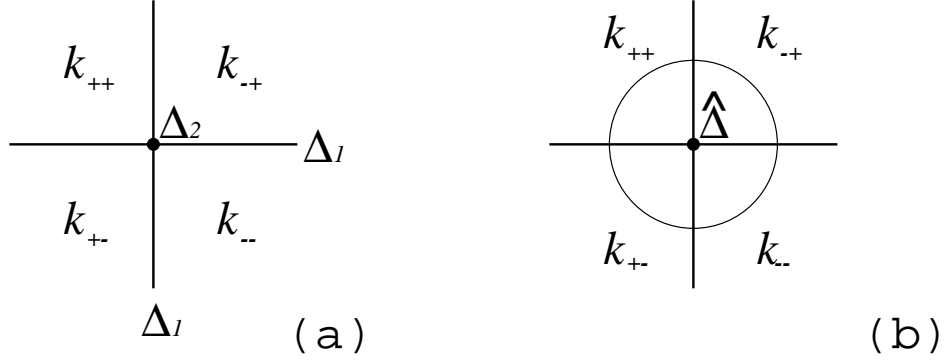


FIGURE 8.

We have a very similar picture in the space of ropes. Indeed, the complement $\widehat{\Delta}$ of $W_L \cup W_R$ in B_ϵ has codimension 2 and a neighbourhood of a generic point of $\widehat{\Delta}$ is pictured on Figure 8 (b). Here the vertical line is formed by ropes which extend to a knot with one double point to the left of A and the horizontal line is formed by ropes which extend with a double point to the right of B . The ropes that extend to genuine knots are labelled by the type of their knot extensions: k_{++} , k_{+-} , k_{--} and k_{-+} .

A circle around $\widehat{\Delta}$ represents the zero element in $H_1(B_\epsilon)$ (which is equal to $\pi_1(B_\epsilon)$ as $\pi_1(B_\epsilon)$ is abelian), however it might not represent the zero element in $H_1(W_L \cup W_R)$. Recall that $W_L \cup W_R$ is a suspension on the space of knots (that are subject to some length restriction). So $H_1(W_L \cup W_R)$ is a free abelian group, generated by non-zero elements of K . Identifying the knots $k_{\pm, \pm}$ with the corresponding generators of $H_1(W_L \cup W_R)$ we see that a circle around $\widehat{\Delta}$ defines, up to sign, the element $k_{++} - k_{+-} + k_{--} - k_{-+}$ in $H_1(W_L \cup W_R)$; and so $k_{++} - k_{+-} + k_{--} - k_{-+} \in \widehat{K}$ is sent to 0 in $\pi_1(B_\epsilon)$.

It remains to show that if the knots k_{++}, k_{+-}, k_{--} and k_{-+} are found as knot types in the neighbourhood of some point of the codimension-2 stratum in Δ , they appear in the same cyclic order as types of knot extensions near some generic point of $\widehat{\Delta}$ when $\epsilon > 2$.

Here, for once, we will make use of round knots. Take a round knot with 2 double points x_1 and x_2 and let k_{++} , k_{+-} , k_{--} and k_{-+} be the resolutions of its singularities. Consider a segment of the knot that connects x_1 and x_2 and choose a point on this segment. If we take this point to be the infinity in S^3 , we get a long knot f such that there are no double points to the left of the smaller value of

the parameter t_1 that corresponds to x_1 and to the right of the larger value of the parameter t_2 that corresponds to x_2 . Let a be slightly larger than t_1 and b slightly smaller than t_2 . We can deform our knot in such a way that $f(a) = A$, $f(b) = B$ and that it is an extension of a rope of length less than $1 + \epsilon$ for any $\epsilon > 2$, see Figure 9. Clearly, this provides us with a rope in $\widehat{\Delta}$ in whose neighbourhood the knot extensions have types k_{++}, k_{+-}, k_{--} and k_{-+} . This proves Theorem 1.2.



FIGURE 9.

5. THE SPACE OF ALL ROPES

Here we construct an explicit deformation retraction of the space B_∞ onto the tight rope. The deformation $\delta_T : B_\infty \times [0, 1] \rightarrow B_\infty$ is the composition of the following transformations:

Similarly to the proof of Theorem 1.3, first we cut the rope at the value of the parameter T , i.e. consider the embedding $r : [0, T] \rightarrow \mathbf{R}^3$. The next step is a homothety with the centre at A :

$$(x, y, z) \rightarrow \frac{(x, y, z)}{|r(T)|}.$$

After this we rotate \mathbf{R}^3 around A so that the “free end of the rope”, i.e. the point $\frac{r(T)}{|r(T)|}$ is moved to B . The rotation $R(r, T)$ is determined from the condition that the derivative $\frac{dR(r, t)}{dt}(T)$ is an infinitesimal rotation around the axis $r(T) \times \frac{dr}{dt}(T)$ of magnitude $|r(T) \times \frac{dr}{dt}(T)| \cdot |r(T)|^{-2}$ and such that $R(r, 0) = \text{Id}$.

It is a straightforward check that δ_T is a continuous deformation retraction with δ_1 being the identity map and δ_0 — the map to the tight rope.

6. TIGHTENING THE ROPES

The proof of Theorem 1.4 resembles in spirit the proofs of Lemma 3.5 and Theorem 1.3: we construct an explicit deformation retraction $B_{\epsilon'} \rightarrow B_\epsilon$ for any $0 < \epsilon < \epsilon' \leq 2$. In other words, we will show how to tighten all ropes in $B_{\epsilon'}$ simultaneously.

First of all let us introduce some notation: $l(r)$ will stand for the length of the rope $r = (r_x(t), r_y(t), r_z(t))$, and $l_x(r)$ and $l_{yz}(r)$ are the lengths of paths $(r_x(t), 0, 0)$ and $(0, r_y(t), r_z(t))$ respectively.

The retraction consists of two steps. First we reduce $l_{yz}(r)$ by squeezing \mathbf{R}^3 to the line AB with the help of some continuous function $h_{r,T}$ with the arguments $r \in B_{\epsilon'}$ and $T \in [0, 1]$:

$$(x, y, z) \rightarrow (x, y h_{r,T}, z h_{r,T}),$$

(compare this with the proof of Lemma 3.5). The second step reduces $l_x(r)$; the deformation of the space of ropes in this case is also induced by a family of deformations of \mathbf{R}^3 . Here they are of the form

$$(x, y, z) \rightarrow (x \phi_{r,T}(x), y, z),$$

where $\phi_{r,T}(x)$ is a family of C^1 -smooth monotonic functions $\mathbf{R} \rightarrow \mathbf{R}$ which depends continuously on parameters r and T .

The function $h_{r,T}$ is chosen as follows. Let $\psi_\tau : B_{\epsilon'} \times [0, 1] \rightarrow B_{\epsilon'}$ be the deformation given by

$$\{r, \tau\} \rightarrow (r_x, (1 - \tau)r_y, (1 - \tau)r_z).$$

It is clear that for any $r \in B_{\epsilon'}$, apart from the tight rope, $l_{yz}(\psi_\tau(r))$ and $l(\psi_\tau(r))$ are decreasing functions of τ and for all ropes $l_x(\psi_\tau(r))$ does not depend on τ . Notice that the function $l_{yz}(\psi_\tau(r))$ is linear in τ and continuous in r , and $\lim_{\tau \rightarrow 1} l_{yz}(\psi_\tau(r)) = 0$ for any $r \in B_{\epsilon'}$. So for all ropes apart from the tight rope we can define $f_1(r)$ as the minimal value of τ such that $l_{yz}(\psi_\tau(r)) \leq \frac{\epsilon}{2}$ and $f_2(r)$ as the minimal value of τ such that $l_{yz}(\psi_\tau(r)) + l_x(\psi_\tau(r)) \leq l(r)$. Now set $f(r) = 1$ if r is the tight rope and $f(r) = \max\{f_1(r), f_2(r)\}$ otherwise. The continuity of $f_1(r)$ and $f_2(r)$ away from the tight rope follows from the linearity of $l_{yz}(\psi_\tau(r))$. It can also be checked directly that as r tends to the tight rope, $f_1(r)$ tends to 0 and $f_2(r)$ tends to 1. (In fact, f_1 is identically zero in some neighbourhood of the tight rope.) Thus, $f(r)$ is also a continuous function $B_{\epsilon'} \rightarrow [0, 1]$ and $f(r) = 1$ if and only if r is the tight rope. Finally, we define $h_{r,T}$ as

$$h_{r,T} = 1 - Tf(r).$$

If $H : B_{\epsilon'} \rightarrow B_{\epsilon'}$ denotes the map

$$r \rightarrow (r_x, r_y h_{r,1}, r_z h_{r,1})$$

it is clear from the above construction that $l_{yz}(H(r)) \leq \frac{\epsilon}{2}$ and $l_{yz}(H(r)) + l_x(H(r)) \leq l(r)$ for all r .

The second step is more involved.

Recall that in the proof of the Proposition 3.2 we defined for each rope r a subset $A(r) \in AB$. Let $Z(r)$ be the complement of $A(r) \cap [AB]$ in $[AB]$. In other words, $Z(r)$ is the subset of interval $[AB]$ formed by such points x that $(p \circ r)^{-1}(x)$ consists of only one point at which $\frac{dr_x}{dt}$ is not zero. The key point in what follows is that for a short rope $Z(r)$ is a non-empty open subset of $[0, 1]$ so it has non-zero Lebesgue measure.

The total length of the projection of r onto AB can be written as a sum $l_x(r) = l_A(r) + l_Z(r)$, where $l_A(r)$ and $l_Z(r)$ are lengths of the parts of $p(r)$ that lie in $A(r)$ and $Z(r)$ respectively. Clearly, $l_Z(r)$ is just the Lebesgue measure of $Z(r)$.

Suppose that we have a family of C^1 -smooth monotonic functions $\phi_{r,T} : \mathbf{R} \rightarrow \mathbf{R}$ which depends continuously on parameters $r \in B_{\epsilon'}$ and $T \in [0, 1]$ and satisfies the following conditions:

- (a) $\phi_{r,T}(0) = 0$ and $\phi_{r,T}(1) = 1$ for all r and T ;
- (b) $\frac{d}{dx}\phi_{r,T}(x) < 1$ for all $x \in A(r)$ and any $T \in (0, 1]$;
- (c) $\frac{d}{dx}\phi_{r,1}(x) \leq \frac{\epsilon}{4l_A(r)}$ for any $x \in A(r)$.

Then we can define a homotopy

$$\Phi_T : H(B_{\epsilon'}) \times [0, 1] \rightarrow B_{\epsilon'}$$

which is given by

$$\{r(t), T\} \rightarrow (\phi_{r,T}(r_x(t)), r_y(t), r_z(t)).$$

Lemma 6.1. *The homotopy Φ_T is well-defined and $\Phi_T(H(B_{\epsilon})) \subset B_{\epsilon}$ for any T .*

Proof. The condition (a) above means that Φ_T takes ropes to ropes; so in order to show that Φ_T is well-defined we need to check that it does not increase the lengths of ropes too much.

It is clear that the Lebesgue measure of $A(r) \cap [AB]$ is equal to $(1 - l_Z(r))$. Let $n(x)$ be the number of inverse images of $p \circ r$ at $x \in AB$. The function $n(x)$ is finite almost everywhere on AB and $n(x) - 1$ is nonnegative on $A(r)$. Integrating $n(x)$ over $A(r)$ we obtain $l_A(r)$, and the sum of integrals,

$$I(r) = \int_{A(r) \cap [AB]} (n(x) - 1) dx + \int_{A(r) \setminus (A(r) \cap [AB])} n(x) dx,$$

is equal to $l_A(r) - 1 + l_Z(r)$. The condition (b) above implies that

$$I(\Phi_T(r)) \leq I(r)$$

so

$$l_x(\Phi_T(r)) = l_A(\Phi_T(r)) + l_Z(\Phi_T(r)) \leq l_A(r) + l_Z(r) = l_x(r).$$

Now recall that for any $r \in B_{\epsilon'}$ the sum of lengths of the projections $l_x(H(r)) + l_{yz}(H(r))$ is less than or equal to $l(r)$. In particular, for all $r \in H(B_{\epsilon'})$ we have $l_x(r) + l_{yz}(r) < 1 + \epsilon'$. It is easy to see Φ_T does not change $l_{yz}(r)$, so the length of $\Phi_T(r)$ is bounded by $l_x(r) + l_{yz}(r) < 1 + \epsilon'$ and this means that Φ_T is well-defined.

Similarly, the inequality $l_x(r) + l_{yz}(r) < 1 + \epsilon$ holds for all $r \in H(B_\epsilon)$ and for all such ropes $l(\Phi_T(r))$ is bounded by $1 + \epsilon$. This means that $\Phi_T(H(B_\epsilon)) \subset B_\epsilon$. \square

Lemma 6.2. *The map $\Phi_1 \circ H : B_{\epsilon'} \rightarrow B_\epsilon$ is a homotopy equivalence.*

Proof. Recall that $l_{yz}(r) \leq \frac{\epsilon}{2}$ for any $r \in H(B_{\epsilon'})$. The condition (c) implies that

$$l_A(\Phi_1(r)) \leq l_A(r) \cdot \frac{\epsilon}{4l_A(r)} = \frac{\epsilon}{4},$$

hence the length of the rope $\Phi_1(r)$ is bounded by

$$l_A(\Phi_1(r)) + l_Z(\Phi_1(r)) + l_{yz}(\Phi_1(r)) \leq \frac{\epsilon}{4} + 1 + \frac{\epsilon}{2} < 1 + \epsilon.$$

So Φ_T deforms $H(B_{\epsilon'})$ into B_ϵ . Moreover, for any T the map Φ_T takes $H(B_\epsilon)$ into B_ϵ and thus $\Phi_1 \circ H$ is a homotopy equivalence. \square

To finish the proof we need to construct the family of functions $\phi_{r,T}$.

Let $\mathfrak{A} \subset B_{\epsilon'} \times AB$ be the subset

$$\mathfrak{A} = \{r, u \mid r \in B_{\epsilon'}, u \in AB, u \in A(r)\}.$$

Recall that in the proof of Lemma 3.5 we have constructed a distance function on the spaces $B_\epsilon \times AB$. For each rope r define a function $g_r(x)$ to be the infimum of the distance from (r, x) to the subset \mathfrak{A} . It is clear that the family of functions $g_r(x)$ is continuous both in x and r . Notice that $g_r(x) = 0$ for any $x \in A(r)$ and $g_r(x) > 0$ for any $x \in Z(r)$. This implies, in particular, that for any short rope r the integral $\int_0^1 g_r(y) dy$ is greater than zero.

Let $\tilde{\phi}_{r,T}(x)$ be equal to

$$\frac{\int_0^x 1 + Tg_r(y) dy}{\int_0^1 1 + Tg_r(y) dy}.$$

For any $x \in A(r)$

$$(*) \quad \frac{d}{dx} \tilde{\phi}_{r,T} = \left(\int_0^1 1 + T g_r(y) dy \right)^{-1} < 1,$$

so if we set

$$\beta(r) = \max \left(0, \left[\frac{4l_A(r)}{\epsilon} - 1 \right] \cdot \left[\int_0^1 g_r(y) dy \right]^{-1} \right)$$

and $\phi_{r,T} = \tilde{\phi}_{r,T\beta(r)}$ the conditions (a), (b) and (c) are satisfied.

Indeed, (a) follows straight from the definitions and (b) is a direct consequence of (*) above. For $x \in A(r)$ we have

$$\frac{d}{dx} \phi_{r,1}(x) = \frac{d}{dx} \tilde{\phi}_{r,\beta}(x) = \left(1 + \beta(r) \int_0^1 g_r(y) dy \right)^{-1} \leq \frac{\epsilon}{4l_A(r)}$$

by (*) and the definition of $\beta(r)$. This verifies (c) and all that is left is to check the continuity of the family $\phi_{r,T}$. This immediately follows from the continuity of $\tilde{\phi}_{r,T}(x)$ and $\beta(r)$ in all variables.

7. FINAL REMARKS AND QUESTIONS

It is certainly interesting that spaces of short ropes provide us with a geometric interpretation of the Grothendieck group of the monoid of knots. However, a stronger question can be asked.

Recall that behind the spaces of short ropes there is a construction of a classifying space. Long knots form an H -space and it is not too hard to modify this H -space to make it associative, i.e. to make it a monoid.

Question 1. *Is the space B_ϵ a classifying space for long knots when $\epsilon \leq 2$?*

For a simplicial monoid M the space ΩBM of loops on the classifying space of M can be considered as a “homological group completion” of M , see [1] or [7]. So there is a chance that the topology of the spaces of short ropes is related to the topology of the space of knots in a rather direct way.

Another question concerns knot invariants. Having got a generic loop L_T in a space of short ropes we can find out what element in \widehat{K} it corresponds to as follows.

Suppose that the knot extensions of L_T have double points to the left of A at the values of the parameter T_i , and let $\delta_i k \in \widehat{K}$ be the formal differences of the types of knot extensions near T_i . Then the loop L_T represents the formal difference $\sum_{T_i} \delta_i k \in \widehat{K}$. (This easily follows from the fact that $W_L \cup W_R$ is a suspension on the space of non-singular knot extensions.) Similarly one can calculate the value of any additive knot invariant on any element of $\pi_1(B_\epsilon)$.

This, however, is not interesting in the sense that we learn nothing new about knots.

Question 2. *Which knot invariants can be defined geometrically on the level of ropes?*

Vassiliev invariants have shown up in our constructions, though in a rather silly way. It would be good to understand if there is a deeper connection. It is interesting that with the help of Vassiliev invariants one can construct groups of knots in a less simple-minded way than taking the Grothendieck group. These are Gusarov’s

groups of n -equivalence classes of knots, see [5] and [8]. Also, cobordism classes of knots form a group, [4]. Clearly, there are homomorphisms from \widehat{K} to all of these groups; it is not clear, however, if these homomorphisms can be interpreted geometrically via ropes.

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